Matemáticas VII.

Preparaduría

Semana #05

1) Diga el valor de la siguiente distribución (10 Ptos).

Parcial ENE-MAR 2007.

a) $\langle (\mathrm{sen}(3x)\,\delta'(x))' \mid \mathrm{sen}(x)\rangle$

Solución

$$\langle (\mathrm{sen}(3x)\,\delta'(x))' \mid \mathrm{sen}(x)\rangle = -\langle \mathrm{sen}(3x)\,\delta'(x) \mid (\mathrm{sen}(x))'\rangle$$

$$= -\langle \delta'(x) \mid \mathrm{sen}(3x)\cos(x)\rangle$$

$$= \langle \delta(x) \mid (\mathrm{sen}(3x)\cos(x))'\rangle$$

$$= \langle \delta(x) \mid 3\cos(3x)\cos(x) - \mathrm{sen}(3x)\,\mathrm{sen}(x)\rangle$$

$$= 3\cos(3\cdot 0)\overbrace{\cos(0)}^{=1} - \cancel{\mathrm{sen}(3\cdot 0)\,\mathrm{sen}(0)}^{=0}$$

$$= 3$$

$$\Rightarrow \boxed{\langle (\mathrm{sen}(3x)\,\delta'(x))' \mid \mathrm{sen}(x)\rangle = 3}$$

b) Hallar los valores de a, b y $c \in \mathbb{R}$ tales que:

$$(e^{-x^2/2}\,\delta'(x-2))' = a\,\delta(x-2) + b\,\delta'(x-2) + c\,\delta''(x-2)$$

Solución

Consideramos

$$\langle (e^{-x^2/2}\,\delta'(x-2))' \mid \varphi(x)\rangle = -\langle e^{-x^2/2}\,\delta'(x-2) \mid \varphi'(x)\rangle$$

$$= -\langle \delta'(x-2) \mid e^{-x^2/2}\varphi'(x)\rangle$$

$$= \langle \delta(x-2) \mid (e^{-x^2/2}\varphi'(x))'\rangle$$

$$= \langle \delta(x-2) \mid -x e^{-x^2/2}\varphi'(x) + e^{-x^2/2}\varphi''(x)\rangle$$

$$= -2e^{-2}\varphi'(2) + e^{-2}\varphi''(2)$$

$$= 2e^{-2}\langle \delta'(x-2) \mid \varphi(x)\rangle + e^{-2}\langle \delta''(x-2) \mid \varphi(x)\rangle$$

$$= \langle 2e^{-2}\delta'(x-2) + e^{-2}\delta''(x-2) \mid \varphi(x)\rangle$$

$$\Rightarrow \boxed{(e^{-x^2/2}\,\delta'(x-2))' = 2e^{-2}\delta'(x-2) + e^{-2}\delta''(x-2)} \Rightarrow \begin{cases} a = 2e^{-2} \\ b = 0 \\ c = e^{-2} \end{cases}$$

3) Parcial (SEP-DIC 2012) (13 Ptos).

Halle la solución causal (A la izquierda de cero) de la ecuación

$$t\,y(t) - 2y'(t) - t\,y''(t) = 2e^{-t}$$

Con condiciones iniciales $y(0)=1$ y $y'(0)=-1$

Solución

$\boxed{u(x) = y(x)\,H(x)}$ $\quad L = tI - 2D - tD^2 \Rightarrow L_{gen} = x\,u(x) - 2u'_{gen}(x) - x\,u''_{gen}(x)$

NOTA: Cambio $x=t$ simplicación en el uso de tablas.

$$u'_{gen}(x) = y'(x)H(x) + y(x)(H(x))'$$
$$= y'(x)H(x) + y(x)\,\delta(x)$$

$$\langle y(x)\delta(x)|\varphi(x)\rangle = \langle \delta(x)|y(x)\varphi(x)\rangle = \overbrace{y(0)}^{=1}\varphi(0) = \varphi(0) = \langle \delta(x)|\varphi(x)\rangle \Rightarrow y(x)\delta(x) = \delta(x).$$

$$\Rightarrow \boxed{u'_{gen}(x) = y'(x)H(x) + \delta(x).}$$

$$u''_{gen}(x) = y''(x)H(x) + y'(x)(H(x))' + \delta'(x)$$
$$= y''(x)H(x) + y'(x)\delta(x) + \delta'(x).$$

$$\langle y'(x)\delta(x)|\varphi(x)\rangle = \langle \delta(x)|y'(x)\varphi(x)\rangle = y'(0)\varphi(0) = -\varphi(0) = -\langle\delta(x)|\varphi(x)\rangle \Rightarrow y'(x)\delta(x) = -\delta(x)$$

$$\Rightarrow \boxed{u''_{gen}(x) = y''(x)H(x) + \delta'(x) - \delta(x)}$$

$$-x u''_{gen}(x) - 2u'_{gen}(x) + x\,u(x) = -x y''(x)H(x) - x\delta'(x) + x\delta(x) - 2y'(x)H(x) - 2\delta(x) + x y(x) H(x).$$
$$= \underbrace{(-x y''(x) - 2y'(x) + x y(x))}_{2e^{-x}}H(x) - x\delta'(x) + x\delta(x) - 2\delta(x)$$

$$\Rightarrow \underbrace{-x u''_{gen}(x) - 2u'_{gen}(x) + x\,u(x)}_{1} = \underbrace{2e^{-x}H(x) - x\delta'(x) + x\delta(x) - 2\delta(x)}_{2}$$

$\xrightarrow{\mathcal{L}}$

$$1 \Rightarrow \frac{\partial}{\partial z}\left(z^2 U(z)\right) - 2zU(z) - \frac{\partial}{\partial z}(U(z)) = \cancel{2zU(z)} + z^2U'(z) - \cancel{2zU(z)} - U'(z)$$
$$= z^2U'(z) - U'(z) = U'(z)\,[z^2-1]$$

$$2 \Rightarrow 2\cdot\frac{1}{z+1} + \frac{\partial}{\partial z}(z) - \frac{\partial}{\partial z}(1) - 2\cdot(1) = \frac{2}{z+1} - 1$$
$$= \frac{2-z-1}{z+1} = \frac{1-z}{z+1}$$

$$\Rightarrow U'(z)\left[z^2-1\right] = \frac{1-z}{z+1} \Rightarrow U'(z) = \frac{1-z}{(z+1)(z^2-1)}$$

$$1-z = -(z-1)$$

$$z^2-1 = (z-1)(z+1) \Rightarrow U'(z) = \frac{-\cancel{(z-1)}}{(z+1)(z+1)\cancel{(z-1)}} = \frac{-1}{(z+1)^2}$$

$$\Rightarrow \underbrace{-U'(z)}_{①} = \underbrace{\frac{1}{(z+1)^2}}_{②}$$

$$\frac{1}{(z+1)^2} = \frac{1}{(z-(-1))^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad ① \Rightarrow x\,u(x)$$

$$② \Rightarrow H(x)\,e^{-x}\,x \quad \Rightarrow x\,u(x) = x e^{-x} H(x)$$

$$\Rightarrow \boxed{u(x) = e^{-x} H(x)}$$

$$\Rightarrow u(x) = y(x)H(x) \quad \Rightarrow \boxed{y(x) = e^{-x}}$$

$$\Rightarrow \boxed{y(t) = e^{-t}}$$

Comprobación.

$$y(t) = e^{-t}$$

$$y'(t) = -e^{-t}$$

$$y''(t) = e^{-t}$$

$$t\,y(t) - 2y'(t) - t\,y''(t) = \cancel{te^{-t}} + 2e^{-t} - \cancel{te^{-t}} = 2e^{-t}$$

3) Parcial ENE-MAR 2012. (15 Ptos).

Si $w>0$, usar la transformada de Laplace para hallar una función generalizada $u(x)$ que cumple

$$u''_{gen}(x)+w^2u(x) = 2\delta(x)+2\delta\left(x-\frac{\pi}{w}\right)$$

y también $u(x)\equiv 0$ para $x\leq 0$.

Solución

Aplicamos transformada de Laplace directamente.

$$z^2U(z)+w^2U(z) = 2+2e^{-\frac{\pi}{w}z} \Rightarrow U(z)\left[z^2+w^2\right] = 2+2e^{-\frac{\pi}{w}z}$$

$$U(z) = \frac{2+2e^{-\pi/w\, z}}{z^2+w^2} \Rightarrow \boxed{U(z) = \frac{2}{z^2+w^2}+\frac{2e^{-\pi/w\, z}}{z^2+w^2}}$$

$$U(z) = \frac{2}{w}\cdot\frac{w}{z^2+w^2}+\frac{2}{w}\cdot\frac{w}{z^2+w^2}\cdot e^{-\pi/w\, z}$$

$$\xrightarrow{\mathcal{L}^{-1}} u(x) = \frac{2}{w}\,\text{sen}(wx)H(x)+\frac{2}{w}\cdot\text{sen}\left(w\left(x-\frac{\pi}{w}\right)\right)H(x-\pi/w)$$

$$= \frac{2}{w}\,\text{sen}(wx)H(x)-\frac{2}{w}\,\text{sen}(wx)H(x-\pi/w)$$

$$= \frac{2}{w}\,\text{sen}(wx)\left[H(x)-H(x-\pi/w)\right]$$

$$\Rightarrow \boxed{u(x) = \frac{2}{w}\,\text{sen}(wx)\,\mathbb{1}_{[0;\pi/w]}(x)}$$